Journal of Nonlinear Mathematical Physics Volume *, Number * (20**), 1–13 Article

A Haldane–Shastry spin chain of BC_N type in a constant magnetic field

A. ENCISO, F. FINKEL, A. GONZÁLEZ-LÓPEZ †*, and M.A. RODRÍGUEZ*

Departamento de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain
† *E-mail: artemio@fis.ucm.es*

*Received Month *, 200*; Revised Month *, 200*; Accepted Month *, 200**

Abstract

We compute the spectrum of the trigonometric Sutherland spin model of BC_N type in the presence of a constant magnetic field. Using Polychronakos's freezing trick, we derive an exact formula for the partition function of its associated Haldane–Shastry spin chain.

1 Introduction

In 1971, F. Calogero [6] introduced a solvable quantum model describing a system of N particles with two-body interactions depending on the inverse square of the particles' distance. In the same year, B. Sutherland [27] proposed a similar model with an interaction potential of trigonometric type. The importance of these models, which was already apparent from the very beginning, is now widely acknowledged by the theoretical and mathematical physics community. From a mathematical point of view, these models are integrable both in their classical and quantum versions, in the sense that they admit a complete set of integrals of motion. Moreover, they are also exactly solvable, in the sense that their spectrum and eigenfunctions can be expressed in closed form. From a more physical point of view, Calogero–Sutherland (CS) models play an important role in many different fields, like for instance Yang–Mills theories [17], the quantum Hall effect [1], random matrices [26, 28] and fractional statistics [18, 21].

Calogero–Sutherland models were cast into a very elegant mathematical framework by Olshanetski and Perelomov in [20]. They showed that these models are limiting cases of a more general one with a two-body interaction potential of elliptic type, and uncovered their relation to the A_N root system. In fact, these authors also constructed generalizations of the previous models associated with all the classical (extended) root systems, like BC_N.

The extension of CS models to particles with internal degrees (typically interpreted as spin) of freedom was first proposed in the classical case in Ref. [16]. In the quantum case, spin CS models were actively studied in the last decade, both in the A_N and BC_N cases. It turns out that these models inherit the basic properties of their scalar counterparts, namely their integrability and exact-solvability. There are essentially two approaches in the study of spin CS models, namely the supersymmetric formalism [5, 8, 15] and the

Dunkl or exchange operator method [2, 3, 9, 11, 12, 22, 29]. Usually, but not always [24], the interaction of the spins with a constant external magnetic field is not included in the spin CS Hamiltonian.

In 1988, Haldane [19] and Shastry [25] independently introduced a new type of solvable spin chain with long-range position-dependent interactions. The sites of this chain are equidistant points in a circle, and the interaction between the corresponding spins is proportional to the inverse square of their chord distance. Shortly afterwards, Fowler and Minahan [14] showed that this chain is completely integrable by means of Polychronakos's exchange operator formalism [22]. The connection with the Sutherland model, although already noted by Shastry in his paper, was made precise by Polychronakos in [23], using what he called the "freezing trick". The main idea behind this method, which can actually be applied to any spin CS model, is to take the strong coupling constant limit in the Hamiltonian, so that the particles become "frozen" at the equilibrium positions of the scalar part of the potential. In this way one can obtain new spin chains of Haldane–Shastry (HS) type, in which the sites are not necessarily equally spaced. Most of the literature on HS spin chains is devoted to those based on dynamical spin models of A_N type, while their BC_N counterparts have received comparatively less attention. Yamamoto and Tsuchiya proved the integrability of the rational HS chain of BC_N type [30], although they did not compute its spectrum. The trigonometric BC_N spin chain was discussed by Bernard, Pasquier and Serban [4] in the spin 1/2 ferromagnetic case, but only for equally spaced sites. The integrability of the trigonometric/hyperbolic version of this chain was established in [7, 13], although again its spectrum was not computed. In a recent work [10], we have extensively studied the trigonometric BC_N spin chains (both ferromagnetic and antiferromagnetic) for arbitrary spin and without assuming that the sites are equally spaced. The main result in the latter paper is the derivation of a closed-form expression for the partition function of the model, following a method based on Polychronakos's freezing trick.

In this paper we study the effect of the presence of a constant external magnetic field in the trigonometric BC_N Sutherland model for spin 1/2 and in its associated spin chain. This modifies the Hamiltonian of the spin chain by the addition of a term proportional to the projection of the total spin operator along the direction of the magnetic field. It turns out that both the dynamical spin model (if the magnetic field is suitably oriented) and its associated chain remain solvable, although the method used in [10] to compute the spectrum of the dynamical model, based on expressing the Hamiltonian in terms of a commuting family of Dunkl operators, cannot be applied in this case. We shall see, however, that the basis of the Hilbert space constructed in the latter reference to solve the model in the absence of a magnetic field can be slightly modified so that the Hamiltonian of the dynamical model is still triangular. From the spectrum of the dynamical model we shall then evaluate in closed form the partition function of its associated spin chain using the freezing trick.

The paper is organized as follows. In Section 2 we introduce the dynamical spin model and its corresponding Hilbert space. The spectrum of this model is computed in Section 3, by showing that the term due to the magnetic field is triangular in an appropriate modification of the basis constructed in [10]. In Section 4 we define the HS spin chain and compute its partition function using the freezing trick and the results of the previous section.

2 Preliminary definitions

In this section we set up the notation used throughout the paper and define the Hamiltonian of the Sutherland spin model of BC_N type in the presence of a constant external magnetic field. We shall denote by $\mathcal{S}$ the Hilbert space corresponding to the internal degrees of freedom of N identical spin $\frac{1}{2}$ particles. Let

$$\mathcal{B}^{\zeta}_{\mathcal{S}} = \big\{|s\rangle \equiv |s_1,\dots,s_N\rangle \;\big|\; s_i = \,\uparrow\downarrow\big\}\,, \tag{2.1}$$

be a basis of $\mathcal{S}$ whose elements are simultaneous eigenstates of the ζ component of the one-particle spin operators, $O\zeta$ being an arbitrary direction. The corresponding spin permutation and reversal operators S_{ij} and S_i are defined by

$$\begin{aligned}
&S_{ij}|s_1,\dots,s_i,\dots,s_j,\dots,s_N\rangle = |s_1,\dots,s_j,\dots,s_i,\dots,s_N\rangle\,,\\
&S_i|s_1,\dots,s_i,\dots,s_N\rangle = |s_1,\dots,-s_i,\dots,s_N\rangle\,.
\end{aligned} \tag{2.2}$$

We will also use the customary notation $\tilde S_{ij} = S_i S_j S_{ij}$. The operators S_i and S_{ij} generate a multiplicative group isomorphic to the Weyl group of B_N type. Similarly, the coordinate permutation and sign-reversal operators K_{ij} and K_i are defined by

$$\begin{aligned}
&(K_{ij}f)(x_1,\dots,x_i,\dots,x_j,\dots,x_N) = f(x_1,\dots,x_j,\dots,x_i,\dots,x_N)\,,\\
&(K_i f)(x_1,\dots,x_i,\dots,x_N) = f(x_1,\dots,-x_i,\dots,x_N)\,,
\end{aligned}$$

and $\tilde K_{ij} = K_i K_j K_{ij}$. The total permutation and sign reversal operators will be denoted by $\Pi_i \equiv K_i S_i$ and $\Pi_{ij} \equiv K_{ij} S_{ij}$. The multiplicative group generated by K_i and K_{ij} (respectively, Π_i and Π_{ij}) is also isomorphic to the B_N-type Weyl group.

Let us define the antisymmetrizer with respect to the symmetric group generated by Π_{ij} as

$$\widehat\Lambda = \frac{1}{N!}\sum_{i=1}^{N!}\operatorname{sgn}(P_i)\,P_i\,,$$

P_i being an element of this group and $\operatorname{sgn}(P_i)$ its signature. Likewise, we will denote by

$$\widehat\Lambda_\epsilon = 2^{-N}\prod_i(1+\epsilon\,\Pi_i)$$

the symmetrization ($\epsilon = 1$) or antisymmetrization ($\epsilon = -1$) with respect to sign reversals. Here and in what follows all sums and products run from 1 to N, unless otherwise stated. We shall make use of the projection operator

$$\Lambda_\epsilon = \widehat\Lambda\,\widehat\Lambda_\epsilon$$

on states antisymmetric under particle permutations and with parity ϵ under sign-reversals. The Hamiltonian of the spin Sutherland model of BC_N type in a constant external mag-

netic field $\mathfrak{B}$ is given by

$$\begin{aligned} H_\epsilon = &-\sum_i \partial_{x_i}^2 + a\sum_{i\neq j}\Big[\sin^{-2}(x_i-x_j)\,(a+S_{ij}) + \sin^{-2}(x_i+x_j)\,(a+\tilde{S}_{ij})\Big] \\ &+ b\sum_i \sin^{-2}x_i\,(b-\epsilon S_i) + b'\sum_i \cos^{-2}x_i\,\big(b'-\epsilon S_i\big) \\ &- eg\,\mathfrak{B}\cdot\mathbf{\Sigma} + \frac{e^2}{4}(\mathfrak{B}_y^2+\mathfrak{B}_z^2)\sum_i x_i^2\,, \end{aligned} \tag{2.3}$$

where $\mathbf{\Sigma} = (\Sigma^\xi, \Sigma^\eta, \Sigma^\zeta)$ is the total spin operator and $(O\xi, O\eta, O\zeta)$ is an arbitrary system of orthogonal axes. Here we have assumed that the real constants a, b, b' are greater than $\frac{1}{2}$, and we have denoted by g and e the particles' gyromagnetic ratio and electric charge, respectively. The last (diamagnetic) term in Eq. (2.3) has to be dropped to preserve the solvability of H_ϵ. In fact, this term vanishes identically if the magnetic field is parallel to the x axis. Since our main interest is to study the spin chain associated with the Hamiltonian H_ϵ, we have preferred in what follows to drop the diamagnetic term in order to keep the direction of the magnetic field in the spin chain arbitrary.

The above Hamiltonian possesses inverse-square type singularities for $x_i \pm x_j = k\pi$ and $x_i = k\pi/2$, with $k \in \mathbb{Z}$. Since the nature of these singularities prevents the particles from overtaking each other and from crossing the singular hyperplanes $x_i = k\pi/2$, the particles may be regarded as distinguishable, with configuration space

$$\widetilde{C} = \Big\{\mathbf{x} = (x_1,\dots,x_N)\in\mathbb{R}^N \;\Big|\; 0 < x_1 < \cdots < x_N < \frac{\pi}{2}\Big\}\,. \tag{2.4}$$

The Hilbert space of the system can thus be taken as $\mathcal{H} = L_0^2(\widetilde{C})\otimes\mathcal{S}$, with

$$\begin{aligned} L_0^2(\widetilde{C}) = \Big\{ f\in L^2(\widetilde{C}) \;\Big|\; &\exists \overline{\lim_{x_i\pm x_j\to k\pi}} |x_i\pm x_j-k\pi|^{-a}|f(\mathbf{x})|\,, \quad \exists \overline{\lim_{x_i\to 0}}\,|x_i|^{-b}|f(\mathbf{x})|\,, \\ &\exists \overline{\lim_{x_i\to\pi/2}}\,|x_i-\pi/2|^{-b'}|f(\mathbf{x})|\,; \quad k=0,1, \quad 1\le i\neq j\le N\Big\}\,. \end{aligned}$$

The conditions imposed in the definition of $L_0^2(\widetilde{C})$ guarantee the finiteness of $(\psi, H_\epsilon\psi)$ for all $\psi\in\mathcal{H}$. It can be shown that the operator $H_\epsilon:\mathcal{H}\to\mathcal{H}$ is actually equivalent to any of its extensions to spaces of symmetric or antisymmetric functions (with respect to both permutations and sign reversals) in $L_0^2(C)\otimes\mathcal{S}$, where C is the N-cube $(-\frac{\pi}{2},\frac{\pi}{2})^N$ and $L_0^2(C)$ is defined similarly to $L_0^2(\widetilde{C})$. For technical reasons, it is convenient to consider that H_ϵ acts in the Hilbert space

$$\mathcal{H}_\epsilon = \Lambda_\epsilon\big(L_0^2(C)\otimes\mathcal{S}\big) \tag{2.5}$$

of states antisymmetric under permutations and with parity ϵ under sign reversals.

3 Spectrum of the dynamical model

The Hamiltonian (2.3) with $\mathfrak{B} = 0$ was thoroughly studied in [10], and its spectrum was exactly computed. The calculation was based on the fact that when $\mathfrak{B} = 0$ the

Hamiltonian (2.3) is the image of the operator

$$\begin{aligned} H_0' = -\sum_i \partial_{x_i}^2 + a \sum_{i\neq j} \Big[\sin^{-2}(x_i - x_j)\,(a - K_{ij}) + \sin^{-2}(x_i + x_j) + \,(a - \tilde{K}_{ij})\Big] \\ + b \sum_i \sin^{-2}x_i\,(b - K_i) + b' \sum_i \cos^{-2}x_i\,\big(b' - K_i\big) \end{aligned} \tag{3.1}$$

under the mapping (cf. [10])

$$K_{ij} \mapsto -S_{ij}, \qquad K_i \mapsto \epsilon S_i\,. \tag{3.2}$$

The operator H_0' was expressed as a sum of squares of an appropriate set of commuting Dunkl operators J_i, $i = 1, \dots, N$, which preserve a flag $\mathcal{R}_0 \subset \mathcal{R}_1 \subset \cdots$ of finite-dimensional spaces of smooth functions, where

$$\mathcal{R}_k = \big\langle \phi(\mathbf{x}) \exp\big(2\mathrm{i} \sum_j n_j x_j\big) \;\big|\; n_j = -k, -k+1, \dots, k\,, \quad j = 1, \dots, N \big\rangle\,, \tag{3.3}$$

and

$$\phi(\mathbf{x}) = \prod_{i<j} \big|\sin(x_i - x_j)\,\sin(x_i + x_j)\big|^a \prod_i |\sin x_i|^b |\cos x_i|^{b'}\,. \tag{3.4}$$

For each value of k, we constructed a basis $\mathcal{B}_k$ of $\mathcal{R}_k$ such that the matrix representing H_0' in this basis is triangular. Since the closure of $\bigcup\limits_{k=0}^{\infty} \mathcal{R}_k$ is $L_0^2(C)$, this observation immediately yields the spectrum of H_0'. The spectrum of H_ϵ in the absence of magnetic field was then computed by suitably extending the basis $\bigcup\limits_{k=0}^{\infty} \mathcal{B}_k$ to a basis $\mathcal{B}_\epsilon^\zeta$ of the Hilbert space $\mathcal{H}_\epsilon$; see [10] for the details.

The operators S_{ij} and S_i can be expressed in terms of the usual one-particle spin operators $\mathbf{\Sigma}_i = (\Sigma_i^\xi, \Sigma_i^\eta, \Sigma_i^\zeta)$ as follows

$$S_{ij} = 2\mathbf{\Sigma}_i \cdot \mathbf{\Sigma}_j + \frac{1}{2}\,, \qquad S_i = 2\Sigma_i^\xi\,.$$

Hence, if the magnetic field in the Hamiltonian (2.3) is directed along the ξ axis, the term $\boldsymbol{\mathfrak{B}} \cdot \mathbf{\Sigma}$ can be expressed in terms of the spin reversal operators S_i as $\frac{1}{2}\mathfrak{B} \sum_i S_i$, where $\mathfrak{B} = |\boldsymbol{\mathfrak{B}}|$. In this case, the Hamiltonian H_ϵ may be written as

$$\begin{aligned} H_\epsilon = -\sum_i \partial_{x_i}^2 + a \sum_{i\neq j} \Big[\sin^{-2}(x_i - x_j)\,(a + S_{ij}) + \sin^{-2}(x_i + x_j)\,(a + \tilde{S}_{ij})\Big] \\ + b \sum_i \sin^{-2}x_i\,(b - \epsilon S_i) + b' \sum_i \cos^{-2}x_i\,\big(b' - \epsilon S_i\big) - \frac{eg}{2}\,\mathfrak{B} \sum_i S_i\,. \end{aligned} \tag{3.5}$$

Although this Hamiltonian is the image of the operator

$$\begin{aligned} H_\epsilon' = -\sum_i \partial_{x_i}^2 + a \sum_{i\neq j} \Big[\sin^{-2}(x_i - x_j)\,(a - K_{ij}) + \sin^{-2}(x_i + x_j)\,(a - \tilde{K}_{ij})\Big] \\ + b \sum_i \sin^{-2}x_i\,(b - K_i) + b' \sum_i \cos^{-2}x_i\,\big(b' - K_i\big) - \frac{\epsilon e g}{2}\,\mathfrak{B} \sum_i K_i\,, \end{aligned}$$

under the mapping (3.2), the Dunkl operator techniques of Ref. [10] cannot be directly applied to prove its integrability, since it is not clear how to express the operator H'_ϵ in terms of a commuting family of differential-difference operators. Moreover, even if the new term proportional to the magnetic field leaves invariant the spaces $\mathcal{R}_k$, the matrix of H'_ϵ in the basis $\bigcup_{k=0}^{\infty} \mathcal{B}_k$ of $L^2_0(C)$ introduced in Ref. [10] need not be triangular. It is more convenient, therefore, to work directly with the spin Hamiltonian (3.5) and its representation in a slight modification of the spin basis $\mathcal{B}^\zeta_\epsilon$ used in the latter reference.

Let us recall, to begin with, the construction of the basis $\mathcal{B}^\zeta_\epsilon$. To this end, we need to introduce the following notation. Let $\mathbb{N}_0^N$ denote the set of nonincreasing multiindices $n = (n_1, \dots, n_N)$, with $n_i = 0, 1, \dots$ and $n_1 \geq \cdots \geq n_N$. If $n, n' \in \mathbb{N}_0^N$, we shall say that $n \prec n'$ if $n_1 - n'_1 = \cdots = n_{i-1} - n'_{i-1} = 0$ and $n_i < n'_i$. Let f_n denote the function

$$f_n(\mathbf{x}) = \phi(\mathbf{x})\, \mathrm{e}^{2\mathrm{i}\sum_i n_i x_i}\,, \qquad n \in \mathbb{N}_0^N\,,$$

where $\phi(\mathbf{x})$ is given by (3.4). The basis $\mathcal{B}^\zeta_\epsilon$ can be taken as any linearly independent subset of the set

$$\big\{ \Lambda_\epsilon(f_n|s\rangle) \;\big|\; n \in \mathbb{N}_0^N\,,\ |s\rangle \in \mathcal{B}^\zeta_{\mathcal{S}} \,\big\} \tag{3.6}$$

ordered so that $\Lambda_\epsilon(f_n|s\rangle)$ precedes $\Lambda_\epsilon(f_{n'}|s'\rangle)$ if $n \prec n'$. It should be noted that the basis $\mathcal{B}^\zeta_{\mathcal{S}}$ in Eq. (3.6) could actually be replaced by any basis of the spin space $\mathcal{S}$ without changing the triangularity of the Hamiltonian $H_\epsilon\big|_{\mathfrak{B}=0}$. As shown in Ref. [10], a linearly independent subset of the set (3.6) is obtained by imposing the following conditions on the multiindex n and the spin basis element $|s\rangle$:

1. $\#(m) \equiv \operatorname{card}\{i \mid n_i = m\} \leq 2 - \delta_{m0}$ for all $m = 0, 1, \dots$;
2. If $n_i = n_{i+1}$, then $s_i = \uparrow$ and $s_{i+1} = \downarrow$;
3. If $n_N = 0$, then $s_N = \uparrow$.

It is more convenient for our purposes to work a slight modification $\mathcal{B}^\xi_\epsilon$ of the basis $\mathcal{B}^\zeta_\epsilon$, obtained by replacing $\mathcal{B}^\zeta_{\mathcal{S}}$ in the previous construction by the basis

$$\mathcal{B}^\xi_{\mathcal{S}} = \big\{ |\sigma\rangle \equiv |\sigma_1, \dots, \sigma_N\rangle \;\big|\; \sigma_i = \pm 1/2 \,\big\} \tag{3.7}$$

of simultaneous eigenstates of the one-particle spin operators Σ^ξ_i. Conditions 2 and 3 above should accordingly be replaced by

2′. If $n_i = n_{i+1}$, then $\sigma_i = +1/2$ and $\sigma_{i+1} = -1/2$;

3′. If $n_N = 0$, then $\sigma_N = \epsilon/2$.

The last condition is due to the fact that the one-particle spin states

$$|\pm 1/2\,\rangle = \frac{1}{\sqrt{2}}\big(\,|\uparrow\rangle \pm |\downarrow\rangle\big)$$

have parity ϵ under flipping of the ζ component of the spin. Since the operator $\sum_i S_i$ commutes with the projector Λ_ϵ, it is diagonal in the basis $\mathcal{B}_\epsilon^\xi$. Indeed,

$$\Big(\sum_i S_i\Big)\Lambda_\epsilon\big(f_n|\sigma\rangle\big) = \Lambda_\epsilon \sum_i f_n\big(S_i|\sigma\rangle\big) = \Big(\sum_i 2\sigma_i\Big)\Lambda_\epsilon\big(f_n|\sigma\rangle\big) \equiv \lambda(n,\sigma)\Lambda_\epsilon\big(f_n|\sigma\rangle\big)\,.$$

Taking into account the conditions 1, 2′, and 3′ for the modified basis $\mathcal{B}_\epsilon^\xi$, it immediately follows that the eigenvalue $\lambda(n,\sigma)$ can be expressed as

$$\lambda(n,\sigma) = d_+(n,\sigma) - d_-(n,\sigma)\,,$$

where

$$d_\pm(n,\sigma) = \operatorname{card}\big\{i \;\big|\; \#(n_i) = 1 \;\text{ and }\; \sigma_i = \pm 1/2\big\}\,.$$

By the previous remark, the Hamiltonian $H_\epsilon\big|_{\mathfrak{B}=0}$ is still triangular in the modified basis $\mathcal{B}_\epsilon^\xi$, with diagonal elements

$$E_n^0 = \sum_i \big(2n_i + b + b' + 2a(N-i)\big)^2\,, \tag{3.8}$$

cf. Ref [10]. It follows that the complete Hamiltonian (3.5) is triangular in the basis $\mathcal{B}_\epsilon^\xi$, with eigenvalues $E_{n\sigma}$ given by

$$E_{n\sigma} = E_n^0 - \frac{eg}{2}\,\mathfrak{B}\,\lambda(n,\sigma)\,. \tag{3.9}$$

This formula for the spectrum of the Hamiltonian (3.5) will be used in next section to compute the partition functions of the corresponding spin chains.

4 An HS spin chain of BC_N type in a magnetic field

Using Polychronakos's freezing trick [23, 24], one may obtain a Haldane–Shastry spin chain of BC_N type associated with the Hamiltonian (3.5). This technique, thoroughly discussed in [10], consists in taking the large coupling constant limit $a \to +\infty$, while maintaining constant the ratios $\beta \equiv b/a$, $\beta' \equiv b'/a$ and $B \equiv -eg\mathfrak{B}/(16a)$. In this limit, the eigenfunctions of H_ϵ become sharply peaked around a minimum of the potential

$$U(\mathbf{x}) = \sum_{i\neq j}\big(\sin^{-2}(x_i - x_j) + \sin^{-2}(x_i + x_j)\big) + \sum_i (\beta^2 \sin^{-2} x_i + \beta'^2 \cos^{-2} x_i)\,, \tag{4.1}$$

and thus the spin and the dynamical degrees of freedom decouple. It is important to note that there is unique minimum $\mathbf{x}^0 = (x_1^0, \dots, x_N^0)$ of the potential U in the Weyl chamber $\widetilde{C}$, as proved in Ref. [10].

Let

$$\begin{aligned} H_{\mathrm{s}} = -\sum_i \partial_{x_i}^2 + a(a-1)\sum_{i\neq j}\big(\sin^{-2}(x_i - x_j) + \sin^{-2}(x_i + x_j)\big) \\ + b(b-1)\sum_i \sin^{-2} x_i + b'(b'-1)\sum_i \cos^{-2} x_i \end{aligned} \tag{4.2}$$

denote the Hamiltonian of the scalar BC_N Sutherland model. The Hamiltonian h_ϵ of the HS spin chain associated with H_ϵ is defined by

$$\mathsf{h}_\epsilon = \frac{1}{a}\left(H_\epsilon - H_{\mathrm{s}}\right)\Big|_{\mathbf{x}\mapsto\mathbf{x}^0}\,, \tag{4.3}$$

namely

$$\mathsf{h}_\epsilon = \sum_{i\neq j}\Big[\sin^{-2}(x_i^0 - x_j^0)\,(1+S_{ij}) + \sin^{-2}(x_i^0 + x_j^0)\,(1+\tilde{S}_{ij})\Big] \\ + \sum_i \big(\beta\,\sin^{-2}x_i^0 + \beta'\,\cos^{-2}x_i^0\big)(1-\epsilon S_i) + 8B\sum_i S_i\,. \tag{4.4}$$

This Hamiltonian differs from the one in [10] by the last term, which represents the interaction of the spins with a magnetic field along the ξ axis of constant magnitude $-16B/(gq)$. As shown in Ref. [10], Eq. (4.3) and the previous considerations lead to the relation

$$\mathsf{Z}_\epsilon(T) = \lim_{a\to\infty}\frac{Z_\epsilon(aT)}{Z_{\mathrm{s}}(aT)}\,. \tag{4.5}$$

between the partition functions Z_ϵ, Z_ϵ, and Z_{s} of the respective Hamiltonians h_ϵ, H_ϵ, and H_{s}.

We shall now compute the partition function of the spin chain (4.4) in closed form by evaluating the RHS of Eq. (4.5). From Eqs. (3.8) and (3.9) it follows that

$$E_{n\sigma} \simeq a^2E_0 + 8a\Big[\sum_i n_i\big(\overline{\beta}+N-i\big) + B\,\lambda(n,\sigma)\Big]\,, \tag{4.6}$$

where $\overline{\beta}\equiv\frac{1}{2}(\beta+\beta')$, and we have dropped the term independent of a which becomes negligible in the limit $a\to\infty$. The constant $E_0\equiv 4\sum_i(\overline{\beta}+N-i)^2$, which is independent of n and σ, can also be dropped from both partition functions Z_ϵ and Z_{s} without modifying the value of Z_ϵ. With this convention, it was proved in [10] that when $a\to\infty$ the partition function of the scalar Sutherland model (4.2) can be written as

$$Z_{\mathrm{s}}(aT) \simeq \prod_i\Big[1-q^{i\left(\overline{\beta}+N-\frac{1}{2}(i+1)\right)}\Big]^{-1}\,, \tag{4.7}$$

where we have set

$$q\equiv \mathrm{e}^{-8/k_{\mathrm{B}}T}\,.$$

The calculation of $Z_\epsilon(aT)$ is more involved. To perform it, it is convenient to represent the multiindex $n\in\mathbb{N}_0^N$ appearing in (4.6) as

$$n = \big(\overbrace{m_1,\dots,m_1}^{k_1},\overbrace{m_2,\dots,m_2}^{k_2},\dots,\overbrace{m_r,\dots,m_r}^{k_r}\big)\,, \tag{4.8}$$

where $m_1>\cdots>m_r\geq 0$ and $k_i=\#(m_i)$ satisfies $\sum_{i=1}^r k_i = N$. The characterization of $\mathcal{B}_\epsilon^x$ in the previous section implies that $k_i\in\{1,2\}$ and $k_r=1$ if $m_r=0$. We shall

denote by $\widetilde{\mathcal{P}}_N$ the set of partitions $k=(k_1,\dots,k_r)$ of the integer N such that $k_i\in\{1,2\}$. If $k=(k_1,\dots,k_r)\in\widetilde{\mathcal{P}}_N$, we define

$$d(k)=\operatorname{card}\left\{i\;\middle|\;k_i=1\right\}.$$

Since $d(k)=d_+(n,\sigma)+d_-(n,\sigma)$, it follows that

$$\lambda(n,\sigma)=2\,d_+(n,\sigma)-d(k)\,, \tag{4.9}$$

and therefore

$$E_{n\sigma}\simeq 8a\Big[\sum_i n_i(\overline{\beta}+N-i)+B\big(2d_+(n,\sigma)-d(k)\big)\Big]\,, \tag{4.10}$$

where we have used Eq. (4.6) without the inessential ground state energy E_0. After expressing the first sum in terms of m and k as in Ref. [10] we obtain the expression

$$E\simeq 8a\Big[\sum_{i=1}^r m_i\nu_i(k)+B\big(2d_+(n,\sigma)-d(k)\big)\Big]\,, \tag{4.11}$$

where

$$\nu_i(k)=k_i\bigg(\overline{\beta}+N-\frac{k_i+1}{2}-\sum_{j=1}^{i-1}k_j\bigg)\,.$$

The partition function $Z_\epsilon(aT)$ is therefore given by

$$Z_\epsilon(aT)\simeq\sum_{k\in\widetilde{\mathcal{P}}_N}\;\sum_{m_1>\cdots>m_r\geq 0}q^{\sum\limits_{i=1}^r m_i\nu_i(k)}\sum_{|\sigma\rangle}q^{B\,[2d_+(n,\sigma)-d(k)]}\,, \tag{4.12}$$

where the sum over the spins is restricted to those values of $|\sigma\rangle$ such that $\Lambda(f_n|\sigma\rangle)\in\mathcal{B}^\xi_\epsilon$. The latter sum depends essentially on whether $m_r>0$ or $m_r=0$. Indeed:

Case 1: $m_r>0$.

In this case, for a given partition $k\in\widetilde{\mathcal{P}}_N$, $d_+(n,\sigma)$ can take any value in the range $0,\dots,d(k)$. The number of states of the basis $\mathcal{B}^\xi_\epsilon$ for which $d_+(n,\sigma)=\delta$ is given by the combinatorial number $\binom{d(k)}{\delta}$. Hence

$$\begin{aligned}\sum_{|\sigma\rangle}q^{B\,[2d_+(n,\sigma)-d(k)]}&=q^{-Bd(k)}\sum_{\delta=0}^{d(k)}\binom{d(k)}{\delta}q^{2B\delta}=q^{-Bd(k)}\Big(1+q^{2B}\Big)^{d(k)}\\&=\Big(q^B+q^{-B}\Big)^{d(k)},\qquad (m_r>0)\,.\end{aligned} \tag{4.13}$$

Case 2: $m_r=0$.

Note, first of all, that Condition 1 on the basis $\mathcal{B}^\xi_\epsilon$ implies that in this case $k_r=1$. Let us suppose, to begin with, that $\epsilon=1$. By condition $3'$ on the basis $\mathcal{B}^\xi_\epsilon$ the ξ component of the spin of the last particle must be $\sigma_N=+1/2$. Thus $d_+(n,\sigma)$ must be at least 1 in

this case. Since the value of σ_N is fixed, the number of states of the basis $\mathcal{B}_\epsilon^\xi$ for which $d_+(n,\sigma)=\delta$ is now given by $\binom{d(k)-1}{\delta-1}$. Therefore

$$\sum_{|\sigma\rangle} q^{B[2d_+(n,\sigma)-d(k)]} = q^{-Bd(k)}\sum_{\delta=1}^{d(k)}\binom{d(k)-1}{\delta-1}q^{2B\delta} = q^B\Big(q^B+q^{-B}\Big)^{d(k)-1}, \qquad (m_r=0,\,\epsilon=1)\,. \tag{4.14}$$

If, on the other hand, $\epsilon=-1$, the only difference with the case $\epsilon=1$ is that now the ξ component of the last particle's spin is $\sigma_N=-1/2$. Obviously, the value of the sum over the spins can be obtained from Eq. (4.14) by changing the sign of B. We thus have

$$\sum_{|\sigma\rangle} q^{B[2d_+(n,\sigma)-d(k)]} = q^{\epsilon B}\Big(q^B+q^{-B}\Big)^{d(k)-1}, \qquad (m_r=0)\,. \tag{4.15}$$

Inserting Eqs. (4.13) and (4.15) into the formula (4.12) for the partition function we obtain

$$Z_\epsilon(aT)\simeq\sum_{k\in\widetilde{\mathcal{P}}_N}\Bigg[\Big(q^B+q^{-B}\Big)^{d(k)}\sum_{m_1>\cdots>m_r>0} q^{\sum_{i=1}^{r}m_i\nu_i(k)} + \delta_{k_r,1}q^{\epsilon B}\Big(q^B+q^{-B}\Big)^{d(k)-1}\sum_{m_1>\cdots>m_{r-1}>0} q^{\sum_{i=1}^{r-1}m_i\nu_i(k)}\Bigg]. \tag{4.16}$$

It was shown in Ref. [10] that

$$\sum_{m_1>\cdots>m_s>0} q^{\sum_{i=1}^{s}m_i\nu_i(k)} = \prod_{j=1}^{s}\frac{q^{N_j}}{1-q^{N_j}}\,, \tag{4.17}$$

with

$$N_j=\sum_{i=1}^{j}\nu_i=\Big(\sum_{i=1}^{j}k_i\Big)\Big(\overline{\beta}+N-\frac12-\frac12\sum_{i=1}^{j}k_i\Big)\,.$$

From Eqs. (4.16) and (4.17) it follows that the partition function Z_ϵ of the Hamiltonian (3.5) satisfies

$$Z_\epsilon(aT)\simeq\sum_{(k_1,\dots,k_r)\in\widetilde{\mathcal{P}}_N}\Big(q^B+q^{-B}\Big)^{d(k)}\Big(\prod_{j=1}^{r-1}\frac{q^{N_j}}{1-q^{N_j}}\Big)\Big(\frac{q^{N_r}}{1-q^{N_r}}+\delta_{k_r,1}\frac{q^{\epsilon B}}{q^B+q^{-B}}\Big). \tag{4.18}$$

Substituting (4.7) and (4.18) into (4.5) we finally obtain the following expression for the partition function of the Haldane–Shastry spin chain (4.4):

$$\mathsf{Z}_\epsilon(T)=\prod_i\Big[1-q^{i\left(\overline{\beta}+N-\frac12(i+1)\right)}\Big]\sum_{(k_1,\dots,k_r)\in\widetilde{\mathcal{P}}_N}\Big(q^B+q^{-B}\Big)^{d(k)}\Big(\prod_{j=1}^{r-1}\frac{q^{N_j}}{1-q^{N_j}}\Big) \times\Big(\frac{q^{N_r}}{1-q^{N_r}}+\delta_{k_r,1}\frac{q^{\epsilon B}}{q^B+q^{-B}}\Big). \tag{4.19}$$

It can be seen that all the denominators $1 - q^{N_j}$ appearing in this formula are included as factors in the first product. Similarly, the denominator in the last term is always canceled by the factor $\left(q^B + q^{-B}\right)^{d(k)}$ (indeed, $d(k) \geq 1$ for $k_r = 1$). Therefore, as should be the case for a finite system, the partition function Z_ϵ can be written as a finite sum of terms of the form $d_{\mathsf{e}}\, q^{\mathsf{e}}$, where $8\mathsf{e}$ is an eigenvalue of the spin chain Hamiltonian h_ϵ and d_{e} its corresponding degeneracy.

The formula (4.19) for the partition function Z_ϵ reduces to that found in Ref. [10] in the absence of magnetic field. We also note that the spin chain Hamiltonians h_+ and h_- are no longer isospectral when $B \neq 0$, although their spectra are obviously related by the mapping $B \mapsto -B$.

Acknowledgments

We would like to dedicate this article to Professor Francesco Calogero in his 70th anniversary. This work was partially supported by the DGI under grant no. BFM2002–02646. A.E. acknowledges the financial support of the Spanish MEC through an FPU scholarship.

References

[1] Azuma H and Iso S, Explicit relation of the quantum Hall effect and the Calogero–Sutherland model, *Phys. Lett.* **B331** (1994), 107–113.

[2] Basu-Mallick B, Spin-dependent extension of Calogero–Sutherland model through anyon-like representations of permutation operators, *Nucl. Phys.*, **B482** (1996), 713–730.

[3] Bernard D, Gaudin M, Haldane F D M and Pasquier V, Yang–Baxter equation in long-range interacting systems, *J. Phys. A: Math. Gen.* **26** (1993), 5219–5236.

[4] Bernard D, Pasquier V and Serban D, Exact solution of long-range interacting spin chains with boundaries, *Europhys. Lett.* **30** (1995), 301–306.

[5] Brink L, Turbiner A and Wyllard N, Hidden algebras of the (super)Calogero and Sutherland models, *J. Math. Phys.* **39** (1998), 1285–1315.

[6] Calogero F, Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* **12** (1971), 419–436.

[7] Caudrelier V and Crampé N, Integrable N-particle Hamiltonians with Yangian or reflection algebra symmetry, *J. Phys. A: Math. Gen.* **37** (2004), 6285–6298.

[8] Desrosiers P, Lapointe L and Mathieu P, Supersymmetric Calogero–Moser–Sutherland models and Jack superpolynomials, *Nucl. Phys.* **B606** (2001), 547–582.

[9] Dunkl C F, Orthogonal polynomials of types A and B and related Calogero models, *Commun. Math. Phys.* **197** (1998), 451–487.

[10] Enciso A, Finkel F, González-López A and Rodríguez M A, Haldane–Shastry spin chains of BC_N type, hep-th/0406054, *Nucl. Phys.* **B**, in press.

[11] Finkel F, Gómez-Ullate D, González-López A, Rodríguez M A and Zhdanov R, A_N-type Dunkl operators and new spin Calogero–Sutherland models, *Commun. Math. Phys.* **221** (2001), 477–497.

[12] Finkel F, Gómez-Ullate D, González-López A, Rodríguez M A and Zhdanov R, New spin Calogero–Sutherland models related to B_N-type Dunkl operators, *Nucl. Phys.* **B613** (2001), 472–496.

[13] Finkel F, Gómez-Ullate D, González-López A, Rodríguez M A and Zhdanov R, On the Sutherland model of B_N type and its associated spin chain, *Commun. Math. Phys.* **233** (2003), 191–209.

[14] Fowler M and Minahan A, Invariants of the Haldane–Shastry $SU(N)$ chain, *Phys. Rev. Lett.* **70** (1993), 2325–2328.

[15] Ghosh P K, Super-Calogero–Moser–Sutherland systems and free super-oscillators: a mapping, *Nucl. Phys.* **B595** (2001), 519–535.

[16] Gibbons J and Hermsen T, A generalisation of the Calogero–Moser system, *Physica* **11D** (1984), 337–348.

[17] Gorsky A and Nekrasov N, Hamiltonian systems of Calogero type, and 2-dimensional Yang–Mills theory, *Nucl. Phys.* **B414** (1994), 213–238.

[18] Ha Z N C, Exact dynamical correlation functions of Calogero–Sutherland model and one-dimensional fractional statistics, *Phys. Rev. Lett.* **73** (1994), 1574–1577.

[19] Haldane F D M, Exact Jastrow–Gutzwiller resonating-valence-bond ground state of the spin-1/2 antiferromagnetic Heisenberg chain with $1/r^2$ exchange, *Phys. Rev. Lett.* **60** (1998), 635–638.

[20] Olshanetsky M A and Perelomov A M, Quantum integrable systems related to Lie algebras, *Phys. Rep.* **94** (1983), 313–404.

[21] Polychronakos A P, Non-relativistic bosonization and fractional statistics, *Nucl. Phys.* **B324** (1989), 597–622.

[22] Polychronakos A P, Exchange operator formalism for integrable systems of particles, *Phys. Rev. Lett.* **69** (1992), 703–705.

[23] Polychronakos A P, Lattice integrable systems of Haldane–Shastry type, *Phys. Rev. Lett.* **70** (1993), 2329–2331.

[24] Polychronakos A P, Exact spectrum of SU(n) spin chain with inverse-square exchange, *Nucl. Phys.* **B419** (1994), 553–566.

[25] Shastry B S, Exact solution of an $S = 1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions, *Phys. Rev. Lett.* **60** (1988), 639–642.

[26] Simons B D, Lee P A and Altshuler B L, Matrix models, one-dimensional fermions, and quantum chaos, *Phys. Rev. Lett.* **72** (1994), 64–67.

[27] Sutherland B, Exact results for a quantum many-body problem in one dimension, *Phys. Rev. A* **4** (1971), 2019–2021.

[28] Taniguchi N, Shastry B S and Altshuler B L, Random matrix model and the Calogero–Sutherland model: A novel current-density mapping, *Phys. Rev. Lett.* **75** (1995), 3724–3727.

[29] Yamamoto T, Multicomponent Calogero model of B_N-type confined in a harmonic potential, *Phys. Lett.* **A208** (1995), 293–302.

[30] Yamamoto T and Tsuchiya O, Integrable $1/r^2$ spin chain with reflecting end, *J. Phys. A: Math. Gen.* **29** (1996), 3977–3984.